A Homotopy Algorithm for Approximating Geometric Distributions by Integrable Systems ⋆

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In the geometric theory of nonlinear control systems, the notion of a distribution and the dual notion of codistribution play a central role. Many results in nonlinear control theory require certain distributions to be integrable. Distributions (and codistributions) are not generically integrable and, moreover, the integrability property is not likely to persist under small perturbations of the system. Therefore, it is natural to consider the problem of approximating a given codistribution by an integrable codistribution, and to determine to what extent such an approximation may be used for obtaining approximate solutions to various problems in control theory. In this note, we concentrate on the mathematical problem of approximating a given codistribution by an integrable codistribution. We present an algorithm for approximating an m-dimensional nonintegrable codistribution by an integrable one using a homotopy approach. The method yields an approximating codistribution that agrees with the original codistribution on an m-dimensional submanifold E_0 of $\mathbb{R}^n$.

Key words: Nonlinear Control Theory, Geometric Distribution, Frobenius Integrability, Differential Forms, Homotopy.

Introduction

In the geometric theory of nonlinear control systems, the notion of a distribution and its dual, codistribution, play a central role. One need only inspect the standard books by Isidori [7] and Nijmeijer and van der Schaft [10] to see that a fair number of theorems require certain distributions to be integrable. Interesting problems of this sort include feedback linearization and observer linearization. Another problem of interest deals with checking the flatness of the Riemannian metric given by the kinetic energy of a robot manipulator [3].

A k-dimensional distribution D is said to be integrable if there are local coordinates $(x_1, \dots, x_n)$ in which $D = \operatorname{span}\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\}$. The classical Frobenius theorem states that a distribution is integrable if and only if it is both involutive and of constant dimension. An m-dimensional codistribution I is called integrable if there are local coordinates $(x_1, \dots, x_n)$ such that $I = \operatorname{span}\{\, \mathrm{d}x_1, \dots, \mathrm{d}x_m\}$. It is well known that a distribution D is integrable if and only if the codistribution $I := D^{\perp}$ (i.e., the codistribution that annihilates D) is integrable.

It is clear that distributions (and codistributions) are not generically integrable and, moreover, the integrability of a distribution will not likely persist under small perturbations of the system. Therefore, it is natural to consider the problem of approximating a given distribution by an integrable distribution, and to determine to what extent such an approximation may be used for obtaining approximate solutions to various problems in control theory. In this note, we concentrate on the mathematical problem of approximating a given codistribution by an integrable codistribution.

A difficulty that arises immediately is that, no matter what method of approximation is used, the achievable accuracy is bounded away from zero. This is due to the fact that the set of integrable distributions forms a closed set in the collection of distributions. For explicit lower bounds on the size of the approximation error, see, e.g., [1,2,6].

The problem of approximating a given distribution (or codistribution) by an integrable one has been considered before in the context of feedback linearization of an affine nonlinear system $\dot{x} = f(x) + g(x)u$. In particular, Krener [9] considered approximation of nonintegrable distributions by integrable ones up to a highest possible order in a neighborhood of an equilibrium point of the system. The papers [5,6,11] deal with approximating nonintegrable distributions by integrable ones about the equilibrium manifold of the system. More

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recently [1] and [2] considered approximations of one-dimensional codistributions by integrable ones on a contractible set containing an equilibrium point of the system.

In this paper we present an algorithm for approximating an m-dimensional nonintegrable codistribution by an integrable one using a homotopy approach. The method yields an approximating codistribution that agrees with the original codistribution on an m-dimensional submanifold E_0 of $\mathbb{R}^n$. Our method for approximating codistributions by integrable ones differs from the methods cited above in many respects. For example, the present approach allows one to choose E_0 is an essentially *arbitrary* manner. In contrast, the method in [11] is such that the approximating distribution agrees with the original one on the m-dimensional *equilibrium* manifold. The papers [1,2] apply only to the case $m = 1$ and provide an approximating codistribution on a contractible region of $\mathbb{R}^n$. When restricted to the case $m = 1$, the method of the present paper allows more general (e.g., not contractible) regions on which the approximation is constructed. While both [1] and the present paper use homotopy approaches to construct approximating codistributions, the present paper uses a more general type of homotopy. Moreover, both [1] and [2] require construction of an approximate integrating factor for the 1-form spanning the original codistribution as a preliminary step in the construction of an approximating codistribution. In the present paper, an approximating codistribution is constructed directly, without prior construction of an approximate integrating factor.

We anticipate that the method presented in this paper can be applied to many problems in nonlinear control theory including approximate feedback linearization, approximate observer linearization, and the approximation of Riemannian metrics by flat metrics.

1 Approximating 1-forms by exact forms

This section is preparatory for the following sections. Given a 1-form ω, we consider the problem of finding an exact 1-form $\mathrm{d}\theta$ that may serve as an approximation to ω.

The classical result of Poincaré states that on a simply connected region every closed form is exact. In the course of proving this result, one usually constructs a homotopy operator H that maps k-forms to $k-1$-forms and that satisfies

$$\omega = \mathrm{d}H\omega + H\,\mathrm{d}\omega. \tag{1}$$

From this it follows that ω is exact when it is closed. Now, in the case that ω

is not closed, one may see this formula as a decomposition of ω into an exact form, $\mathrm{d}H\omega$, and an error term.

Assume that ω is defined on the simply connected manifold M. Assume furthermore that

$$\phi_t : M \to M, \quad t \in [0,1],$$

is a homotopy, i.e., $(t,x) \mapsto \phi_t(x)$ is a smooth map, with the following properties:

(i) $\phi_1 : x \mapsto x$,
(ii) $\phi_0 : x \mapsto x^0$,
(iii) For all $s,t : \phi_s \circ \phi_t = \phi_{st}$.

Then we may define a homotopy operator H depending on ϕ_t by

$$\theta(x) = H\left(\sum_i w_i \,\mathrm{d}x_i\right) = \int_0^1 \sum_i w_i(\phi_t(x)) \frac{\mathrm{d}}{\mathrm{d}t}\phi_t(x)\,\mathrm{d}t.$$

Using this formula, if ω is closed on M then $\mathrm{d}\theta = \omega$. But how good is the approximation of $\mathrm{d}\theta$ if ω is not closed? This is a difficult question to answer in full generality and will necessarily involve the error term $H\,\mathrm{d}\omega$. For our purposes we can state the following. Defining the vector field

$$X(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} \phi_t(x), \tag{2}$$

we have $\langle\, \mathrm{d}\theta - \omega, X\rangle = 0$. In other words, as functionals on an n-dimensional vector space, $\mathrm{d}\theta$ and ω agree in (at least) one direction. Without more information about ω, there is no guarantee that $\mathrm{d}\theta$ and ω will agree on any other vector fields besides X.

Theorem 1 *Let M be a contractible manifold. Let $\phi_t : M \to M$ be a homotopy on M, as above, and $X(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1}\phi_t(x)$. Let ω be a 1-form on M. There is a unique exact form $\mathrm{d}\theta$ on M such that*

$$\langle\, \mathrm{d}\theta, X\rangle = \langle \omega, X\rangle\,. \tag{3}$$

PROOF. We derive a formula for θ. Define $F = \langle \omega, X\rangle$ and assume that θ satisfying equation (3) exists. Then it satisfies $X(\theta) = F$ so that

$$F(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} \theta(\phi_t(x)), \quad \text{for all } x \in M.$$

Evaluating this formula at $\phi_s(x)$, we get

$$F(\phi_s(x)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} \theta(\phi_t\phi_s(x)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} \theta(\phi_{st}(x))$$
$$= s\frac{\mathrm{d}}{\mathrm{d}s}\theta(\phi_s(x))$$

from which it follows that

$$\boxed{\theta(x) = \theta(x^0) + \int_0^1 \frac{F(\phi_s(x))}{s}\,\mathrm{d}s.} \qquad (4)$$

Clearly $\mathrm{d}\theta$ is uniquely determined if it exists. It is straightforward to show that θ defined by (4) satisfies (3). $\square$

This result shows that an exact form on M is entirely determined by its value on the vector field X. One can imagine that the approximation is not accurate in the direction of other vector fields but at x^0 the two forms do agree, namely

$$\mathrm{d}\theta(x^0) = \omega(x^0).$$

We give a proof of this fact in the following section.

Note that Theorem 1 does *not* imply that an exact form is fully determined once it is known on a particular vector field. For example, take $Z = \frac{\partial}{\partial x_1}$ and respectively $\omega = \mathrm{d}(x_1 + \xi(x_2, \ldots, x_m))$ for any function ξ, then $\langle \omega, Z \rangle = 1$.

2 Approximating codistributions by integrable systems

In this section we generalize the ideas of the previous section to the problem of approximating an m-dimensional codistribution by an integrable system.

Let M be a manifold foliated by m-dimensional manifolds E_α and let $\phi_t : M \to M$ be a homotopy having the following properties:

(i) ϕ_1 is the identity map on M,
(ii) If x^1, x^2 belong to the leaf E_α then, for each t, $\phi_t(x^1)$ and $\phi_t(x^2)$ belong to the same leaf,
(iii) ϕ_0 is a projection on a particular leaf E_0, i.e., $\phi_0 \circ \phi_0 = \phi_0$ and $\mathrm{Im}(\phi_0) = E_0$,
(iv) For all $s, t : \phi_s \circ \phi_t = \phi_{st}$.

As before, define the vector field X associated with the homotopy ϕ_t by $X(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1}\phi_t(x)$. The restriction $\omega|_N$ of a 1-form ω to a submanifold $N \subset M$ is the 1-form on N such that for every vector v tangent to N we have $\langle \omega, v\rangle = \langle \omega|_N, v\rangle$. One of the main results of the paper is the following theorem.

Theorem 2 *Let M be a manifold foliated by m-dimensional manifolds E_α and let ϕ_t be a homotopy on M as described above. Let $\omega_1, \ldots, \omega_m$ be 1-forms on M that are linearly independent when restricted to E_0. Then, in a neighborhood of E_0, there are unique 1-forms $\bar\omega_1, \ldots, \bar\omega_m$ such that*

(i) $\operatorname{span}\{\bar\omega_1, \ldots, \bar\omega_m\}$ *is integrable,*
(ii) $\langle \bar\omega_i, X\rangle = \langle \omega_i, X\rangle$, *where* $X(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1}\phi_t(x)$,
(iii) $\bar\omega_i|_{E_\alpha} = \omega_i|_{E_\alpha}$ *on all leaves* E_α.

PROOF. Let $F_i = \langle \omega_i, X\rangle$ and write $F = (F_1, \ldots, F_m)^T$. Introduce local coordinates $x = (y_1, \ldots, y_{n-m}, z_1, \ldots, z_m)$ near E_0 such that the leaves of the foliation are locally given by y_i = constant and, in particular, E_0 is given by $y = 0$.

Let $\omega_i|_{E_\alpha} = \sum_j g_{ij}(y, z)\,\mathrm{d}z_j$. By assumption, $g = (g_{ij})$ is invertible near E_0. Without loss of generality, we assume that $g = \mathrm{Id}$. Otherwise simply start with the generators $g^{-1}\omega$ for I.

From the first condition, we may assume that $\bar\omega_i = \sum_j B_{ij}\,\mathrm{d}\theta_j$ for some functions $\theta_1, \ldots, \theta_m$. The second and third condition require that, for every point x near E_0 and all $0 \le s \le 1$,

$$\begin{aligned} sB(\phi_s(y,z))\frac{\partial}{\partial s}\theta(\phi_s(y,z)) &= F(\phi_s(y,z)),\\ B(\phi_s(y,z))\frac{\partial}{\partial z}\theta(\phi_s(y,z)) &= \mathrm{Id}. \end{aligned} \tag{5}$$

Now, fix $y \neq 0$ and define the functions $\Theta(s, z) = \theta(\phi_s(y, z))$. Eliminating B from equations (5) gives, for $i = 1, \ldots, n$,

$$\frac{\partial \Theta_i}{\partial s} - \sum_{j=1}^{m} \frac{F_j(\phi_s(y,z))}{s}\frac{\partial \Theta_i}{\partial z_j} = 0.$$

This is a system of n identical first order PDEs in the functions $\Theta_i(s, z)$. Each solution Θ will be constant on the characteristic curves

$$\boxed{\begin{aligned} &\frac{\mathrm{d}z}{\mathrm{d}s} = -\frac{F(\phi_s(y,z))}{s}\\ &z(0) = z^0. \end{aligned}} \tag{6}$$

We get a particular solution by specifying the initial condition

$$\Theta(0, z^0) = z^0 \tag{7}$$

and from this choice it is clear that the functions θ_i are independent functions near E_0. The solution to (6) is given by $z(s) = \psi(s, (y, z^0))$ and satisfies $\psi(st, (y, z^0)) = \psi(s, \phi_t(y, z^0))$. Hence, for $(0, z^0) \in E_0$, $\psi(1, (0, z^0)) = \psi(1, \phi_0(y, z^0)) = \psi(0, (y, z^0)) = z^0$. But then $\frac{\partial}{\partial z^0}\psi(1, (0, z^0)) = \mathrm{Id}$ and hence the function $z^0 \mapsto \psi(1, (y, z^0))$ is invertible, for y near 0. Let $\theta(y, z)$ be the implicit function satisfying $z = \psi(1, (y, \theta(y, z)))$. The approximation is given by the system $\operatorname{span}\{\,\mathrm{d}\theta_1, \ldots, \,\mathrm{d}\theta_m\}$. The matrix B of functions is determined by (5).

Now we show uniqueness of $B\,\mathrm{d}\theta$. Assume that $B'\,\mathrm{d}\theta'$ is a different solution. Then θ' is also constant on the integral curves (6). Both θ and θ' form a set of m independent functions when restricted to E_0 and thus the map $\theta' \mapsto \theta$ is invertible in a neighborhood of E_0. Then

$$\begin{aligned}
B\,\mathrm{d}\theta &= \left(\frac{\partial\theta}{\partial z}\right)^{-1} \mathrm{d}\theta \\
&= \left(\frac{\partial\theta}{\partial\theta'}\frac{\partial\theta'}{\partial z}\right)^{-1} \left(\frac{\partial\theta}{\partial\theta'}\right) \mathrm{d}\theta' = \left(\frac{\partial\theta'}{\partial z}\right)^{-1} \mathrm{d}\theta' \\
&= B'\,\mathrm{d}\theta'.
\end{aligned}$$

□

The proof of this theorem provides an algorithm for constructing an integrable codistribution approximating a given codistribution $\operatorname{span}\{\omega_1, \ldots, \omega_m\}$. The following steps are involved:

(i) Choose a foliation of M by m-dimensional manifolds E_α and a corresponding homotopy ϕ_t (as defined in the beginning of this section). Introduce local coordinates $(y_1, \ldots, y_m, z_1, \ldots, z_{n-m})$ near E_0 such that the leaves of the foliation are locally given by $y_i =$ constant, and in particular $E_0 : y = 0$.

(ii) Define the vector function $F = (\langle\omega_1, X\rangle, \ldots, \langle\omega_m, X\rangle)$ where $X(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1}\phi_t(x)$.

(iii) Find the solution $\psi(s, y, z^0)$ to the system of ODE given by (6).

(iv) In the solution put $s = 1$ and find the implicit vector function θ satisfying $z = \psi(1, y, \theta(y, z))$. The integrable codistribution $\bar{I}$ is given by $\operatorname{span}\{\,\mathrm{d}\theta_1, \ldots, \,\mathrm{d}\theta_m\}$. The approximation error is $\left(\omega_i - \sum_j B_{ij}\,\mathrm{d}\theta_j\right)$ where $B = \left(\frac{\partial\theta}{\partial z}\right)^{-1}$.

This algorithm requires the solution of a system of nonlinear ODEs followed

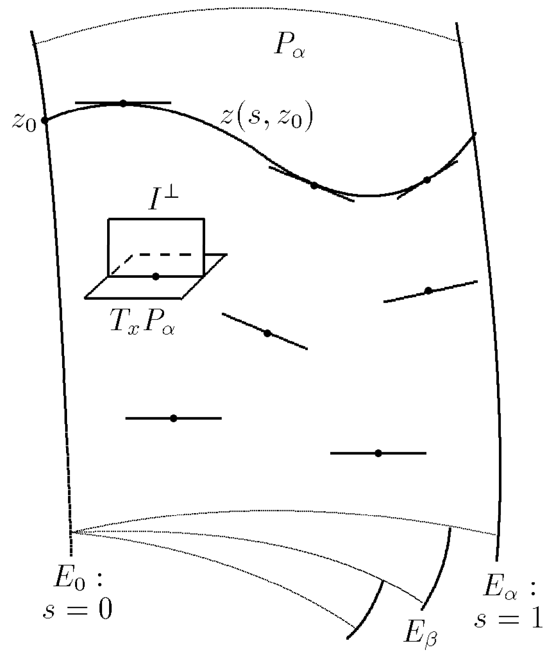

Fig. 1. A line field on the page P_α

by the calculation of an implicit function. These can be accomplished numerically. Note that even in the case of exact integrability one must solve a system of ODEs for the characteristic curves. Note also that it follows from the uniqueness of $\bar{\omega}_i, i = 1, \ldots, m$, that if $\mathrm{span}\{\omega_1, \ldots, \omega_m\}$ is integrable then $\bar{\omega}_i = \omega_i$, i.e. the algorithm recovers the original system of generators.

With a bit of imagination, Theorem 2 may be regarded as a generalization of Theorem 1: take $m = 0$ and replace integrability of a codistribution by the exactness of a single 1-form ω. Condition 3 of Theorem 2 is automatically satisfied since each E_α is a point of M.

In the sequel we will use the manifolds $P_\alpha = \{\phi_s(x) \ : \ x \in E_\alpha, \ s \in [0, 1]\}$ which we will refer to as *pages* (this name is suggested by Figure 1). Then Theorem 2 provides a unique set of 1-forms $\bar{\omega}_1, \ldots, \bar{\omega}_m$ for which

$$\bar{\omega}_i|_{P_\alpha} = \omega_i|_{P_\alpha}, \quad i = 1, \ldots, m$$

and $\mathrm{span}\{\bar{\omega}_1, \ldots, \bar{\omega}_m\}$ is integrable. A geometric way to understand this result is as follows. A curve $\gamma(s)$ in M (i.e. a characteristic curve) on which θ is constant necessarily satisfies (see Figure 1)

$$\gamma'(s) \in T_{\gamma(s)} P_\alpha \cap I^\perp_{\gamma(s)}.$$

Since this intersection is 1-dimensional, γ is an integral of a line field. The equations for this line field are exactly (6).

Observe that Theorem 2 provides an integrable codistribution $\bar{I} = \text{span}\{\bar{\omega}_1, \ldots, \bar{\omega}_m\}$ approximating a codistribution $I = \text{span}\{\omega_1, \ldots, \omega_m\}$. It may seem that the approximation depends on the choice of generators $\omega_1, \ldots, \omega_m$ of I. The following Theorem shows that this is not the case. Namely, the approximating codistribution $\bar{I}$ is uniquely determined by I and the choice of pages P_α.

Theorem 3 *Let M be a manifold foliated by m-dimensional manifolds E_α and let ϕ_t be a homotopy on M as described above. Let I be an m-dimensional codistribution on M which restricted to E_0 is also m-dimensional. Then, in a neighborhood of E_0, there is a unique codistribution $\bar{I}$ such that*

(i) $\bar{I}$ *is integrable.*
(ii) $\bar{I}\big|_{P_\alpha} = I|_{P_\alpha}$ *on all pages P_α.*

PROOF. Let $\omega_1, \ldots, \omega_m$ be a choice of generators for I and let $\bar{\omega}_1, \ldots, \bar{\omega}_m$ be the unique 1-forms of Theorem 2. Change to new generators $B\omega$ for I, where B is a nonsingular matrix of functions. Now, the codistribution generated by $B\bar{\omega}$ is integrable and moreover

$$(B\bar{\omega})|_{P_\alpha} = B|_{P_\alpha}\, \bar{\omega}|_{P_\alpha} = B|_{P_\alpha}\, \omega|_{P_\alpha} = (B\omega)|_{P_\alpha} = \overline{B\omega}\Big|_{P_\alpha}.$$

The uniqueness part of Theorem 2 guarantees that $B\bar{\omega} = \overline{B\omega}$ and hence that $\bar{I} = \text{span}\{\bar{\omega}_1, \ldots, \bar{\omega}_m\}$ is well defined. □

A special case occurs if M is $(m+1)$-dimensional. In that case, each $\bar{\omega}_i = \omega_i$, since each page P_α is $(m+1)$-dimensional. Of course, it is clear that every m-dimensional codistribution on an $(m+1)$-dimensional manifold is integrable.

Under fairly general conditions, the codistribution $\bar{I}$ that approximates I agrees with I on E_0.

Proposition 4 *Assume $\Sigma_\alpha T_x P_\alpha = T_x M$ for all $x \in E_0$. Then $\omega_i(x) = \bar{\omega}_i(x)$ for all $x \in E_0$.*

PROOF. Let Z be a tangent vector at $x \in E_0$. Then there exist tangent vectors $Z_{\alpha_i} \in T_x P_{\alpha_i}$ such that $Z = Z_{\alpha_1} + \cdots + Z_{\alpha_k}$. By Theorem 2, $\omega_j(x)|_{P_{\alpha_i}} = \bar{\omega}_j(x)|_{P_{\alpha_i}}$. It follows from this that $\langle \omega_j(x), Z \rangle = \langle \bar{\omega}_j(x), Z \rangle$. Since Z was arbitrary, the conclusion follows. □

An important property of this algorithm is that, if the original codistribution I contains an integrable subsystem J, then the approximating codistribution $\bar{I}$ also contains J. We have the following result.

Proposition 5 *Let $\bar{I}$ be the approximation of I given by Theorem 3. Then* $\mathrm{d}\theta \in I \Rightarrow \mathrm{d}\theta \in \bar{I}$.

PROOF. Choose generators $\omega_1, \ldots, \omega_m$ for I and $\mathrm{d}\theta_1, \ldots, \mathrm{d}\theta_m$ for $\bar{I}$ such that

$$\omega_i = \mathrm{d}\theta_i + \Omega_i$$

where $\Omega_i|_{P_\alpha} = 0$. Furthermore, assume that θ is a function on M such that $\mathrm{d}\theta = \sum_i \lambda_i \omega_i \in I$. Thus,

$$\mathrm{d}\theta = \sum_i \lambda_i \,\mathrm{d}\theta_i + \Omega, \quad \text{where } \Omega := \sum_i \lambda_i \Omega_i.$$

Note that $\Omega|_{P_\alpha} = 0$ for each P_α.

If each function $\lambda_i = 0$ then $\mathrm{d}\theta$ is constant on each page P_α and hence θ is constant on M.

So assume without loss of generality that $\lambda_1 \neq 0$. Define $J = \mathrm{span}\{\,\mathrm{d}\theta,\, \mathrm{d}\theta_2, \ldots, \mathrm{d}\theta_m\}$. Then J is completely integrable and $J|_{P_\alpha} = I|_{P_\alpha}$ for every page P_α. Uniqueness of $\bar{I}$ (Theorem 3) implies $J = \bar{I}$. Hence $\mathrm{d}\theta \in J$. □

3 Examples

We discuss two examples of approximating a given control system by a feedback linearizable system. In both cases, the obstruction to feedback linearization of the system is due to the nonintegrability of a distribution. Therefore, the problem may be approached naturally in two steps. First approximate the distribution by an integrable distribution, using the algorithm described above. For the sake of comparison, we include expressions for the error. Second construct a feedback transformation that approximately linearizes the system.

Various questions remain at this point, among which: 1) How to use the free choice of homotopy to obtain as good an approximation as possible; 2) How to deal with the general case, which leads to a chain of possibly non-integrable distributions? 3) Is the method employed in the second step a smart way to find the transformation? We plan to address these questions in a seperate publication.

Example 6 In [6] the problem to find a feedback linearizable system approximating the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = ax_1 u + x_3 \\ \dot{x}_3 = u \end{cases} \tag{8}$$

is considered. Define the vector fields $f = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2}, g = ax_1 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}$. Then (8) is not feedback linearizable since $\mathrm{span}\{g, [f,g]\} = \omega^\perp$, where

$$\omega = \mathrm{d}x_1 - \frac{ax_1}{1-ax_2}\,\mathrm{d}x_2 + \frac{(ax_1)^2}{1-ax_2}\,\mathrm{d}x_3 \tag{9}$$

is not integrable.

I. For E_0 we choose the *equilibrium manifold*, which consists of all points that can serve as an equilibrium point under the appropriate input. In our case the equilibrium manifold is given by $x_2 = x_3 = 0$. Let $x = (z, y_1, y_2)$ and introduce the foliation of $\mathbb{R}^3$ given by $E_{(y_1,y_2)} : y_1 = \text{constant}, y_2 = \text{constant}$. We choose the homotopy $\phi_t(z, y_1, y_2) = (z, ty_1, ty_2)$ and thus $X = y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}$. Therefore $F = \langle \omega, X \rangle = \frac{-azy_1 + (az)^2 y_2}{1-ay_1}$ and $\omega|_{E_{(y_1,y_2)}} = \mathrm{d}z$. The equations (6) become

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}s} &= \frac{azy_1 - (az)^2 y_2}{1 - say_1} \\ z(0) &= z^0, \end{aligned} \tag{10}$$

which has for solution $z(s) = \frac{z^0}{1 - asy_1 + sa^2 z^0 y_2}$. Put $s = 1$ and solve for z_0. We obtain:

$$\theta(z, y_1, y_2) := z_0 = \frac{z(1-ay_1)}{1-a^2 zy_2}.$$

At this moment, we have found the approximating system, $\mathrm{span}\{\mathrm{d}\theta\}$. To obtain the approximation errors, we calculate, using (5), the function $B = \left(\frac{\partial\theta}{\partial z}\right)^{-1}$. The approximation of ω by an integrable 1-form (expressed in the original coordinates) is given by

$$\begin{aligned} \bar{\omega} = \frac{(1-a^2 x_1 x_3)^2}{1-ax_2}\,\mathrm{d}\left(\frac{x_1(1-ax_2)}{1-a^2 x_1 x_3}\right) = \\ \mathrm{d}x_1 - \frac{ax_1(1-a^2 x_1 x_3)}{1-ax_2}\,\mathrm{d}x_2 + (ax_1)^2\,\mathrm{d}x_3. \end{aligned} \tag{11}$$

The approximation error is

$$\omega - \bar{\omega} = -\frac{a^3 x_1^2 x_3}{1 - a x_2}\,\mathrm{d}x_2 + \frac{a^3 x_1^2 x_2}{1 - a x_2}\,\mathrm{d}x_3.$$

Note that $\omega(x) = \bar{\omega}(x)$ at points $x \in E_0$. Moreover, the error vanishes when $a = 0$, which agrees with the fact that ω is integrable when $a = 0$.

II. Now we proceed to use the above approximation to find a suitable feedback transformation. Define $(\theta_1, \theta_2, \theta_3) = (\theta, \mathcal{L}_f\theta, \mathcal{L}_f^2\theta)$. Then $(x_1, x_2, x_3) \to (\theta_1, \theta_2, \theta_3)$ defines a state transformation. The control system may be expressed in the new coordinates by

$$\begin{aligned}
\dot{\theta}_1 &= \theta_2 + u(-x_1^2 x_2 a^3 + O(a^4))\\
\dot{\theta}_2 &= \theta_3 + u(-2x_1^2 x_2 a^3 + O(a^4))\\
\dot{\theta}_3 &= F(\theta_1, \theta_2, \theta_3) + u(1 - 3x_2 a + O(a^2)).
\end{aligned}$$

In these coordinates, a possible approximation is by the linear system $\dot{\theta}_1 = \theta_2, \dot{\theta}_2 = \theta_3, \dot{\theta}_3 = v$. Here we have applied the feedback $v = u + F(\theta_1, \theta_2, \theta_3)$.

We note that the approximate feedback linearizable system found in [6] is of the same order in a as we have found here.

Example 7 Consider the contact system $I = \mathrm{span}\{\omega_1, \omega_2\}$, $\omega_1 = \mathrm{d}z_1 - a z_2\,\mathrm{d}y_1$, $\mathrm{d}z_2 - b y_2\,\mathrm{d}y_1$. This system arises in studying feedback linearization of the control system

$$\begin{cases}
\dot{z}_1 = y_1 + a z_2 u_1\\
\dot{y}_1 = u_1\\
\dot{z}_2 = y_2 + b y_2 u_1\\
\dot{y}_2 = u_2.
\end{cases} \tag{12}$$

I. We choose for E_0 the plane $y_1 = y_2 = 0$ and for the homotopy $\phi_t(z_1, z_2, y_1, y_2) = (z_1, z_2, t y_1, t y_2)$ and thus $X = y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}$. Hence $F = \begin{pmatrix} -a z_2 y_1 \\ -b y_1 y_2 \end{pmatrix}$ and the characteristic equations are

$$\frac{\mathrm{d}z_1}{\mathrm{d}s} = a z_2 y_1, \qquad \frac{\mathrm{d}z_2}{\mathrm{d}s} = s b y_1 y_2.$$

Solutions are given by $z_1 = \frac{ab}{6} s^3 y_1^2 y_2 + s a y_1 z_2^0 + z_1^0$, $\quad z_2 = \frac{b}{2} s^2 y_1 y_2 + z_2^0$. Put

$s = 1$ and solve for z_1^0, z_2^0. We obtain:

$$\begin{aligned}\theta_1 &= z_1 - ay_1z_2 + \frac{ab}{3}y_1^2y_2 \\ \theta_2 &= z_2 - \frac{b}{2}y_1y_2.\end{aligned} \tag{13}$$

The approximating system is $\bar{I} = \mathrm{span}\{\,\mathrm{d}\theta_1,\ \mathrm{d}\theta_2\}$. To obtain the approximation error first calculate $B = \left(\frac{\partial\theta}{\partial z}\right)^{-1} = \begin{pmatrix} 1 & ay_1 \\ 0 & 1 \end{pmatrix}$ and then that

$$\begin{aligned}\omega_1 - \bar{\omega}_1 &= -\frac{ab}{6}y_1(y_2\,\mathrm{d}y_1 - y_1\,\mathrm{d}y_2) \\ \omega_2 - \bar{\omega}_2 &= -\frac{b}{2}(y_2\,\mathrm{d}y_1 - y_1\,\mathrm{d}y_2).\end{aligned}$$

II. To find the approximate feedback linearizable system to (12), define the states $\theta_3 = \mathcal{L}_f\theta_1 = y_1 - ay_1y_2, \theta_4 = \mathcal{L}_f\theta_2 = y_2$. The system expressed in the θ-coordinates is given by

$$\begin{aligned}\dot{\theta}_1 &= \theta_3 + u_1(-\frac{1}{3}aby_1y_2) + u_2(\frac{1}{3}aby_1^2) \\ \dot{\theta}_2 &= \theta_4 + u_1(\frac{1}{2}by_2) + u_2(-\frac{1}{2}by_1) \\ \dot{\theta}_3 &= u_1(1 - ay_2) \\ \dot{\theta}_4 &= u_2.\end{aligned}$$

Now approximate the system by the linear system $\dot{\theta}_1 = \theta_3, \dot{\theta}_2 = \theta_4, \dot{\theta}_3 = v_1, \dot{\theta}_4 = v_2$, where $v_1 = u_1(1 - ay_2), v_2 = u_2$.

Conclusion

In this note, we studied the problem of approximating a given codistribution by an integrable codistribution. We presented an algorithm for approximating an m-dimensional nonintegrable codistribution by an integrable one using a homotopy approach. The method yields an approximating codistribution that agrees with the original codistribution on an m-dimensional submanifold E_0 of $\mathbb{R}^n$. The algorithm was demonstrated on two worked examples.

References

[1] A. Banaszuk and J. Hauser. Approximate feedback linearization: homotopy operator approach. *SIAM J. Control*, 34, 1996. To appear.

[2] A. Banaszuk, A. Święch, and J. Hauser. Least squares integration of one-dimensional codistributions with application to approximate feedback linearization. *MCSS*, 1995. Submitted.

[3] N.S. Bedrossian and M.W. Spong. Feedback linearization of robot manipulators and Riemannian curvature. *Journal of Robotic Systems*, 12(8):541–552, 1995.

[4] H. Flanders. *Differential Forms*. Dover Publications, New York, 1963.

[5] J. Hauser. Nonlinear control via uniform system approximation. *Systems & Control Letters*, pages 145–154, 1991.

[6] J. Hauser and E.C. Gwo. Approximate integration of nonintegrable distributions. In *Proceedings of the 31st Conference on Decision and Control*, pages 1589–1593, Tucson, Arizona, December 1992.

[7] A. Isidori. *Nonlinear Control Systems*. Springer–Verlag, Berlin, second edition, 1989.

[8] W. Kang and A.J. Krener. Extended quadratic controller normal form and dynamic state feedback linearization of nonlinear systems. *SIAM J. Control and Optimization*, 30(6):1319–1337, November 1992.

[9] A.J. Krener. Approximate linearization by state feedback and coordinate change. *Systems & Control Letters*, pages 181–185, December 1984.

[10] H. Nijmeijer and A.J. van der Schaft. *Nonlinear Dynamical Control Systems*. Springer–Verlag, New York, 1990.

[11] Z. Xu and J. Hauser. Higher order approximate feedback linearization about a manifold for multi-input systems. *IEEE Transactions on Automatic Control*, 40(5):833–840, May 1995.